

On the Nature of the Photon

In 1864, the Scottish physicist James Clerk Maxwell undertook the task of combining all that was then known about electric and magnetic phenomena into a single, encompassing theory. Maxwell used as a template the mathematical structure of the well-known theory of elastic media. In this regard, Maxwell followed in the long tradition of applying established mathematical tools to a new problem. Maxwell's original papers are somewhat difficult to read because they do not incorporate the modern vector notation invented by Oliver Heaviside. In this regard, it makes Maxwell's achievement even more compelling as he was able to understand the mathematical structure present in the equations even without the notational support that makes it more evident.

The principal modification that Maxwell made to the existing theories of electric and magnetic phenomena was to add a term that he perceived was necessary due to the symmetry of the equations. It is known as the displacement current, for historical reasons, but the real impact of Maxwell's contribution was that it converted a system of equations that described electrical and magnetic phenomena as separate entities into a single, comprehensive theory of electromagnetism. Indeed, as we shall encounter shortly, the division of phenomena into electric and magnetic components depends on the frame of reference. This makes the formulation inconsistent with the principle of relativity and we shall see subsequently how to remedy that problem.

In large measure, Maxwell's equations form the most successful physical theory ever developed. To be sure, thermodynamics provided the underpinnings for the industrial revolution and quantum electrodynamics has demonstrated unprecedented agreement with high-precision measurements. Nonetheless, electromagnetic phenomena form the heart of all of our modern technology. We generate electrical power and distribute it across the world. We build electrical motors that drive ubiquitous machinery and have mastered the generation of electromagnetic radiation that cooks our food in microwave ovens and allows us to communicate

over vast distances. We use lasers to scan groceries at checkout counters and send down images from satellites orbiting over the planet that enable us to more accurately predict weather patterns. All of these disparate phenomena can be modelled quite precisely with Maxwell's equations.

Additionally, Maxwell's equations have served as a template for essentially all further developments in theoretical physics. As a theory of abstract fields, Maxwell's equations have been extended, generalized and scrutinized in an effort to explain other physical phenomena, leading to the development of the Standard Model of high-energy particle physics and string theory beyond.

2.1. Maxwell's Equations

Maxwell's equations represent a classical field theory that links the existence and time variation of the field quantities to sources: charge distributions and currents. We shall begin by stating Maxwell's equations in a modern, integral form, using the SI system of units. Units are always problematic in dealing with electromagnetic phenomena. As a practical matter, force has the dimension of $(M \cdot L \cdot T^{-2})$, where M is a mass, L is a length and T represents time. The Coulomb force (QE) requires a dimension for the electric field of $(M \cdot L \cdot T^{-2} \cdot Q^{-1})$. There are alternative unit systems that can potentially simplify the equations by suppressing some of the constants that arise but their use is not compelling.

Using vector notation, the equations can be written rather compactly. In their integral form, Maxwell's equations can be written as follows:

$$(2.1) \quad \int_{\partial V} d^2 \mathbf{r}_2 \mathbf{n}(\mathbf{r}_2) \cdot \mathbf{D}(t_2, \mathbf{r}_2) = \int_V d^3 \mathbf{r}_1 \rho(t_1, \mathbf{r}_1),$$

$$(2.2) \quad \int_{\partial V} d^2 \mathbf{r}_2 \mathbf{n}(\mathbf{r}_2) \cdot \mathbf{B}(t_2, \mathbf{r}_2) = 0,$$

$$(2.3) \quad \int_{\partial S} d\mathbf{l} \cdot \mathbf{E}(t_2, \mathbf{r}_2) = -\frac{d}{dt} \int_S d^2 \mathbf{r}_1 \mathbf{n}(\mathbf{r}_1) \cdot \mathbf{B}(t_1, \mathbf{r}_1) \quad \text{and}$$

$$(2.4) \quad \int_{\partial S} d\mathbf{l} \cdot \mathbf{H}(t_2, \mathbf{r}_2) = \int_S d^2 \mathbf{r}_1 \mathbf{n}(\mathbf{r}_1) \cdot \mathbf{J}(t_1, \mathbf{r}_1) + \frac{d}{dt} \int_S d^2 \mathbf{r}_1 \mathbf{n}(\mathbf{r}_1) \cdot \mathbf{D}(t_1, \mathbf{r}_1).$$

Here, we have implicitly assumed that all vectors are three-dimensional. The fields are functions of both time and space. These are quite formidable expressions, so let us spend some time discussing their meaning.

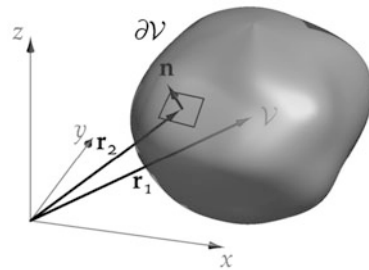
The electric displacement $\mathbf{D}$ is related to the electric field $\mathbf{E}$ by what are known as constitutive relations. In physical media, the presence of an external electric field causes a number of different effects: charge separation in metals and polarization in insulators. For modest field strengths in

most media, the two are simply proportional: $\mathbf{D} = \epsilon \mathbf{E}$, where ϵ is known as the dielectric permittivity. In general, the relationship can be more complex, ϵ may be a tensor, wherein any single component of the electric displacement will depend upon all of the components of the electric field:

$$D_i = \epsilon_{ix}E_x + \epsilon_{iy}E_y + \epsilon_{iz}E_z,$$

Indeed, it is not necessary for $\mathbf{D}$ and $\mathbf{E}$ to be related linearly but we shall defer discussions of that nature to later in the text. Similarly, the magnetic induction $\mathbf{B}$ is often treated as having a linear dependence on the magnetic field $\mathbf{H}$: $\mathbf{B} = \mu \mathbf{H}$, where μ is called the magnetic permeability. In vacuum, ϵ and μ are just constants, usually denoted as ϵ_0 and μ_0 .

FIGURE 2.1. A charge distribution is located within the volume V . Points $\mathbf{r}_1$ lie within the volume. The integral extends over the surface of the volume ∂V . Points $\mathbf{r}_2$ lie on the surface, with local normal $\mathbf{n}$.



In the first equation 2.1, known as Gauss's law, we find that, if we integrate $(d^3\mathbf{r}_1)$ over a (three-dimensional) volume V that contains a distribution of charge ρ (known as the charge density), this will be equal to the two-dimensional $(d^2\mathbf{r}_2)$ integral of the component of the electric displacement $\mathbf{D}$ that is everywhere normal to the surface (∂V) that bounds V . An illustration of such a volume is depicted in figure 2.1. At the point $\mathbf{r}_2$ on the surface of the volume, there is the (outwardly directed) normal $\mathbf{n}$. The component of the displacement parallel to the normal that emerges through the infinitesimal surface element (denoted by the small rectangle) is then summed (integrated) over all such surface elements to obtain the total flux.

There are several important assumptions captured in equation 2.1. First, we have defined a continuous function ρ that defines the charge density within the volume. As all charge ultimately arises from the charges on individual electrons or nuclei, this is clearly an approximation but one which serves generally well for macroscopic systems. We have also assumed that the volume is finite, or mathematically, that the charge density has finite support. If we were to consider a universe with a constant charge extending to infinity, then the charge density integral would diverge, leading to the loss of our ability to make any use of equation 2.1. For the mathematically aware, there is also the implicit assumption that

the surface $\partial\mathcal{V}$ is orientable, meaning that we can define a normal everywhere. Not all manifolds have this property and we are additionally limiting ourselves to surfaces that are reasonably smooth, or at least piecewise smooth. Indeed, most examples found in introductory texts restrict the surfaces to spheres and cylinders.

EXERCISE 2.1. The volume element in spherical coordinates is given by the following:

$$d^3\mathbf{r} = dr d\theta d\varphi r^2 \sin\theta,$$

where r is the radial coordinate, θ is the polar angle measured from the z -axis and φ is the azimuthal angle measured from the x -axis. If we have a charge distribution $\rho(\mathbf{r}) = \rho(r)$ that is independent of the angular variables, i.e., is spherically symmetric, what is the result of the integrating over θ and φ ?

We can note that the right-hand side of equation 2.1 will vanish if the total charge within the volume $\mathcal{V}$ vanishes. This does not mean that the charge density must be everywhere zero, only that there are equal amounts of positive and negative charge within the volume. In any case, it is also true that any charge outside the volume $\mathcal{V}$ does not contribute to the right-hand side integral. Consequently, any charge outside the volume does not add to the net flux (left-hand side) through the surface of the volume.

In many instances, we will be interested in the fields due to just a single charge or a few charges. In those cases, we can still utilize equation 2.1 as a statement of Gauss's law by utilizing the delta function $\delta(x)$ introduced by Dirac. The delta function has the following property:

$$(2.5) \quad \int_a^b dx f(x) \delta(x-c) = f(c),$$

provided that $a \leq c \leq b$. Strictly speaking, the delta function is known to mathematicians as a distribution or a generalized function. Actually, Dirac's contrivance provoked a large amount of work by mathematicians to demonstrate that such a function could even be defined in any sort of sensible fashion. For students, the appearance of a delta function is generally a godsend, evaluating a complex integral is reduced to writing down the value of the integrand at the point $x = c$.

In the case of a single charge, located at the point $\mathbf{r}_0 = (x_0, y_0, z_0)$, the charge density will be given by the following expression:

$$\rho(\mathbf{r}_1) = Q \delta(x_1 - x_0) \delta(y_1 - y_0) \delta(z_1 - z_0) = Q \delta^3(\mathbf{r}_1 - \mathbf{r}_0).$$

Inserting this expression into the charge density integral, we obtain:

$$\int_{\mathcal{V}} d^3\mathbf{r}_1 \rho(\mathbf{r}_1) = Q,$$

provided that $\mathbf{r}_0$ lies within the volume $\mathcal{V}$, otherwise, the integral vanishes.

Turning now to the second Maxwell equation 2.2, we find a similar expression for the magnetic induction $\mathbf{B}$ but, in this case, the integral always vanishes. This result can be taken to mean that there is no magnetic charge.

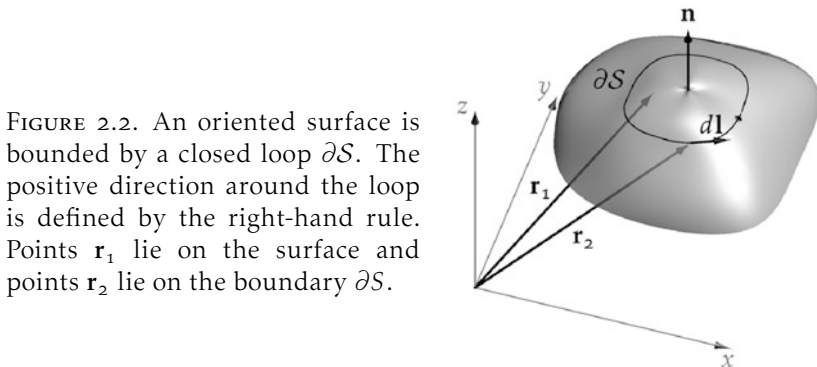


FIGURE 2.2. An oriented surface is bounded by a closed loop ∂S . The positive direction around the loop is defined by the right-hand rule. Points $\mathbf{r}_1$ lie on the surface and points $\mathbf{r}_2$ lie on the boundary ∂S .

The next equation 2.3 is known as Faraday's law. It relates the time rate of change of the magnetic flux through a surface S to the line integral of the electric field around the perimeter ∂S . At some level, equation 2.3 is the basis of our modern technology: the changing magnetic flux through a surface defined by a metal loop gives rise to an electric field (and a current by Ohm's law) within the loop. This is the definition of an electrical generator or a transformer, depending upon the application. If we, instead, consider the induced current to be the source term, equation 2.3 provides the definition of an electric motor. These facts may not be explicit in the rather curious glyphs contained within the mathematics but those outcomes can be obtained nonetheless.

In figure 2.2, we illustrate the relationship between the surface S and its boundary ∂S . Points $\mathbf{r}_1$ within the boundary of an oriented surface, with local normal $\mathbf{n}(\mathbf{r}_1)$. Points $\mathbf{r}_2$ on the boundary of the surface trace a one dimensional path, where locally the direction $d\mathbf{l}$ is positive in a right-handed sense.

The final equation 2.4 was originally proposed by Ampère but subsequently modified by Maxwell, who included the last term on the right-hand side out of concerns of symmetry with the corresponding Faraday's law. In its final form, the Ampère-Maxwell law provides that the current flow and the time rate of change of the electric flux through a surface S is

proportional to the line integral of the magnetic field around the boundary of the surface. It is actually the last term, known historically as the displacement current, that provides the final clue as to the nature of light.

The student, at this point, will most likely not be suddenly awakened to the secrets of the universe. How these equations depict the nature of electromagnetic phenomena is still opaque. In truth, mathematicians have not provided extensive means for solving integral equations, although the integral form can form the basis of numerical, finite element methods. Rather than investigate that option here, we shall follow the path most often used by physicists and convert the integral equations into a series of partial differential equations.

We can make use of the following vector identities, that hold for suitably well-behaved vector fields:

$$(2.6) \quad \int_V d^3\mathbf{r} \nabla \cdot \mathbf{A} = \int_{\partial V} d^2\mathbf{r} \mathbf{n}(\mathbf{r}) \cdot \mathbf{A} \quad \text{and}$$

$$(2.7) \quad \int_S d^2\mathbf{r} \nabla \times \mathbf{A} = \int_{\partial S} d\mathbf{l} \cdot \mathbf{A}.$$

Here, $\mathbf{A}$ represents some vector field and ∇ is a differential operator. The vector $\mathbf{n}$ is the unit normal to the surface ∂V . In Cartesian coordinates, the operator ∇ has the following form:

$$\nabla \equiv \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right].$$

Note that we use the term operator and not vector. Because ∇ is composed of differentials, it really only has meaning when applied to functions. Technically, mathematicians will call the differential operator a 1-form but we will defer this discussion until later. As an example, ∇ applied to a scalar function f results in a vector that we call the *gradient*:

$$(2.8) \quad \nabla f(x, y, z) = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right].$$

It is possible to construct two other mathematical objects from the differential operator. The term $\nabla \cdot \mathbf{A}$ is called the *divergence* of the field $\mathbf{A}$ and is a scalar function. The object $\nabla \times \mathbf{A}$ is called the *curl* of $\mathbf{A}$ and is a vector. We'll discuss more about the mathematical underpinnings of the operator subsequently but, for the present, we can simply state that we interpret

these operations to mean the following:

$$(2.9) \quad \nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad \text{and}$$

$$(2.10) \quad \nabla \times \mathbf{A} = \hat{\mathbf{x}} \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right] + \hat{\mathbf{y}} \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right] + \hat{\mathbf{z}} \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right].$$

Using the vector identities 2.6 and 2.7, we can rewrite Maxwell's equations as a series of differential equations:

$$(2.11) \quad \nabla \cdot \mathbf{D} = \rho,$$

$$(2.12) \quad \nabla \cdot \mathbf{B} = 0,$$

$$(2.13) \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{and}$$

$$(2.14) \quad \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.$$

Here, we have suppressed an integration over an unspecified volume $\mathcal{V}$, assuming that equations 2.11–2.14 are true pointwise, precisely because the volume is not specified. In truth, there is a formidable amount of mathematics underlying our ability to switch back and forth between the differential and integral forms of the Maxwell equations. Undoubtedly, few students will have much appreciation for that fact at the moment and few students will observe that equations 2.11–2.14 represent any sort of real progress in understanding the meaning of Maxwell's equations. Indeed, solving coupled partial differential equations remains a formidable challenge. Nevertheless, a good deal of progress has been made in constructing solutions and, with the advent of modern computers, numerical approaches have extended the applicability to quite complex geometries.

It is nonetheless remarkable that we can derive two separate representations of the Maxwell equations. We have chosen to write them in a fashion that emphasizes the operator nature of both the integration and differentiation operators. We will see ultimately that the two are inverses of one another but the fact that we have two representations means that, if we cannot find a way to solve the integral equation, perhaps we can find a way forward by considering the differential equation. This sort of strategy has been used quite successfully, as we shall see.

EXERCISE 2.2. Use Gauss's law for magnetic fields, equation 2.2, and the divergence identity, equation 2.6, to convert the separate integrations into a single volumetric integration. Show that you can recover equation 2.12.

We can show an important result with modest effort, though. Consider taking the curl of equation 2.13. We can make use of a vector identity:

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - (\nabla \cdot \nabla)\mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}.$$

We then find the following result:

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{E}) &= -\nabla \times \frac{\partial \mathbf{B}}{\partial t} \\ \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} &= -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}), \end{aligned}$$

where in the second step we have assumed that the order of differentiation can be reversed. Let us now make the simplification that we are considering free space and that the constitutive relations between the fields are simply proportional. That is, we assume that $\mathbf{D} = \epsilon_0 \mathbf{E}$ and $\mathbf{B} = \mu_0 \mathbf{H}$. Then we can use equation 2.14 to write

$$\begin{aligned} \nabla \frac{\rho}{\epsilon_0} - \nabla^2 \mathbf{E} &= -\frac{\partial}{\partial t} \left(\mu_0 \mathbf{J} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \right) \\ (2.15) \quad \epsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} &= -\nabla \frac{\rho}{\epsilon_0} - \mu_0 \frac{\partial \mathbf{J}}{\partial t}. \end{aligned}$$

Now in the absence of local sources ρ and $\mathbf{J}$, equation 2.15 is just the homogeneous wave equation for each of the components of the electric field, provided that we identify $\epsilon_0 \mu_0 = 1/c^2$, where c is the velocity of light. Note the particular importance of the displacement current in the establishment of the wave equation. In any case, this was a significant prediction of Maxwell's equations, one which was verified by Heinrich Hertz in a series of experiments that he conducted during the years 1887–1889.¹

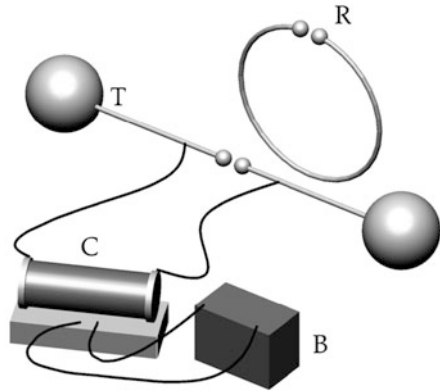
EXERCISE 2.3. Expand the term $\nabla \times \nabla \times \mathbf{A}$ into Cartesian components and show that the vector identity discussed above is correct.

EXERCISE 2.4. Take the curl of the Ampère-Maxwell equation 2.14 and derive a similar wave equation for the magnetic field, using the same constitutive relations for the field.

Hertz's apparatus was quite ingenious, making repeated use of Faraday's law; a basic representation of his early devices is illustrated in figure 2.3. A battery was connected via a switch to a circuit composed of a transformer and capacitor, as depicted in figure 2.4. When the switch is closed, a transient voltage pulse is created in what amounts to an LC circuit. That pulse is amplified to high voltage through the transformer and applied to the two arms of the antenna. If the voltage difference is large enough, a spark leaps the gap between the two arms of the antenna, resulting in a

¹Hertz published "Über sehr schnelle electrische Schwingungen" in the *Annalen der Physik* in 1887.

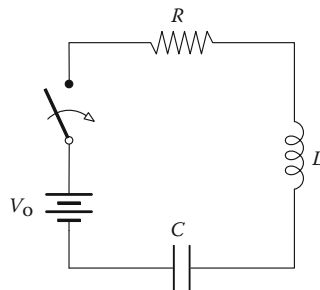
FIGURE 2.3. Hertz's apparatus included a battery (B) connected to an induction coil (C) and thence to a transmitting antenna (T). The receiver (R) was constructed of a metal loop.



large current pulse along the antenna. This current pulse results in the radiation of electromagnetic energy. If a changing magnetic field impinges through the receiver, a current will be generated in the loop. For sufficiently large currents, a spark will leap the gap.

Hertz found that, if he worked in a darkened room and allowed his eyes to adapt sufficiently, that he could readily detect the small sparks emanating from his receiver. He found that the waves could be blocked by metal screens, that the waves bounced from metal screens with an angle of reflection equal to the angle of incidence and, by orienting the receiver in different directions, that the waves were polarized. That is, the electromagnetic fields were well described by the wave equation. Hertz was gratified to confirm the predictions made by Maxwell but saw no particular use for the phenomenon. Commercialization would require the intervention of others like Guglielmo Marconi and Alexander Graham Bell.

FIGURE 2.4. Hertz's transmitter circuit can be modelled as a simple RLC circuit, as depicted at right, where the resistance arises from the finite resistivity of the wires.



EXERCISE 2.5. The equation for the current I flowing in the circuit as a function of time can be written as follows:

$$-L \frac{dI(t)}{dt} - RI(t) - \frac{1}{C} \int_0^t d\tau I(\tau) + V(t) = 0,$$

where $V(t)$ is zero for times before $t = 0$ and V_0 thereafter. The Laplace transform can be used to convert the differential equation into an algebraic equation in the transform variable s .

Use the `LaplaceTransform` function on each of the terms in the equation to derive the transformed equation. Note, use the variable `I1` for the current because `I` is reserved for the imaginary number i . Show that the Laplace transform of the current is given by the following:

$$\tilde{I}(s) = \frac{V_0}{Ls^2 + Rs + 1/C}.$$

Use the `InverseLaplaceTransform` function to demonstrate that if $I(0) = 0$, then

$$I(t) = \frac{V_0}{\sqrt{R^2 - 4L/C}} e^{-\alpha t} [e^{\beta t} - e^{-\beta t}],$$

where

$$\alpha = \frac{R}{2L} \quad \text{and} \quad \beta = \left[\frac{R^2}{4L} - \frac{1}{LC} \right]^{1/2}.$$

The behavior of the circuit depends upon the value of β . If $1/LC$ is larger than $R^2/4L$, then the square root becomes imaginary and the exponentials become oscillatory.

Plot the behavior of the system if $V_0 = 20$ V, $R = 1$ Ω , $L = 3$ mH and $C = 47$ nF for $0 \leq t \leq 1$ ms. What happens if you increase R ?

2.2. Fields and Potentials

Maxwell's equations expressed in their differential form remain just as formidable as the integral form but physicists and mathematicians have developed a number of methods for their solution. One of the most useful is the integral transform, pioneered by Joseph Fourier and Simon Laplace, among others. Mathematically, this strategy is based on the fact that one can define a functional space and a basis of functions within that space. Then, one can uniquely expand *any* function in terms of the basis functions. The Fourier transform of some function $F(x)$ is defined as follows:

$$(2.16) \quad \tilde{F}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx F(x) e^{-ikx}$$

and the inverse transform is defined as follows:

$$(2.17) \quad F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \tilde{F}(k) e^{ikx}.$$

Here we have adopted the physicist's standard notation, wherein an overall factor of $(1/2\pi)$ that is required to normalize the outcome is shared between the forward and inverse transforms. Mathematicians often adopt alternative definitions that avoid the square root and associate the factor with either the forward or inverse transform. Students should be aware that different texts may be using different definitions.

Fourier's inversion theorem can be thus stated as follows:

$$(2.18) \quad \begin{aligned} F(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx} \int_{-\infty}^{\infty} d\zeta F(\zeta) e^{-ik\zeta} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\zeta \int_{-\infty}^{\infty} dk e^{ik(x-\zeta)} F(\zeta). \end{aligned}$$

We note that this can be true only if

$$(2.19) \quad \delta(x - \zeta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ik(x-\zeta)}.$$

We have discovered an integral representation of the Dirac delta function, which will prove useful subsequently.

The utility of applying Fourier transforms to differential equations stems from the following observation:

$$(2.20) \quad \begin{aligned} \frac{dF(x)}{dx} &= \frac{1}{\sqrt{2\pi}} \frac{d}{dx} \int_{-\infty}^{\infty} dk \tilde{F}(k) e^{ikx} \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk (ik) \tilde{F}(k) e^{ikx}. \end{aligned}$$

That is, the x -dependence of the Fourier transform lies solely within the exponential. The transformed function $\tilde{F}$ is a function of the transform variable k and not a function of x . So, taking the derivative, so long as it is possible to exchange the order of integration and differentiation, simply brings down a factor of (ik) . Higher order derivatives are multiplied by powers of (ik) . As a result, differential equations become algebraic equations in the transform domain. Those can be readily solved and all that remains is an (often formidable) integral back into the spatial domain. Physicists often make use of Fourier transforms in the time domain, to remove the time-dependence of a system of equations. In this case, the transform variable ω has the dimension of inverse time, or frequency. One then talks of working in the frequency domain and a specific time-dependence can be recovered by performing the inverse transform integration.

The transforms can also be applied to vector equations. Suppose now that we have a function of the vector $\mathbf{r} = (x, y, z)$. Then, we can transform each component separately, with a (vector) transform variable $\mathbf{k} = (k_x, k_y, k_z)$:

$$\begin{aligned} F(\mathbf{r}) &= \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \int_{-\infty}^{\infty} dk_z \tilde{F}(\mathbf{k}) e^{i(k_x x + k_y y + k_z z)} \\ (2.21) \quad &\equiv \frac{1}{(2\pi)^{3/2}} \int d^3\mathbf{k} \tilde{F}(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}}. \end{aligned}$$

Here, we have illustrated the three dimensional Fourier transform.

EXERCISE 2.6. Consider the function $F(\mathbf{r})$ as defined in equation 2.21. What is the partial derivative with respect to x ? Compute the gradient of the function: $\nabla F(\mathbf{r})$, where

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right).$$

Write your answer in terms of the vector $\mathbf{k}$.

In vacuum, the transformed Maxwell equations (suppressing the integrals over $d\omega d^3\mathbf{k}$) become the following:

$$(2.22) \quad i\epsilon_0 \mathbf{k} \cdot \tilde{\mathbf{E}} = \tilde{\rho},$$

$$(2.23) \quad i\mu_0 \mathbf{k} \cdot \tilde{\mathbf{H}} = 0,$$

$$(2.24) \quad i\mathbf{k} \times \tilde{\mathbf{E}} = i\mu_0 \omega \tilde{\mathbf{H}}, \quad \text{and}$$

$$(2.25) \quad i\mathbf{k} \times \tilde{\mathbf{H}} = \tilde{\mathbf{J}} - i\omega\epsilon_0 \tilde{\mathbf{E}}.$$

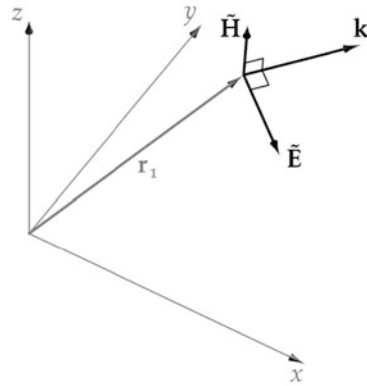
In source-free regions, the first two equations 2.22 and 2.23 tell us that the wave vector $\mathbf{k}$ is perpendicular to both the (transformed) electric $\tilde{\mathbf{E}}$ and magnetic $\tilde{\mathbf{H}}$ fields, as depicted in figure 2.5. The second two equations tell us that $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ are themselves perpendicular. The wave vector $\mathbf{k}$ defines the direction of propagation of the wave. By convention, the polarization of the wave is defined by the direction of the electric field.

EXERCISE 2.7. For a vector function $\mathbf{F}(\mathbf{r}) = (F_x(\mathbf{r}), F_y(\mathbf{r}), F_z(\mathbf{r}))$ show that, in the transform domain, the divergence and curl operators yield the following results:

$$\nabla \cdot \mathbf{F} \Rightarrow (i\mathbf{k}) \cdot \tilde{\mathbf{F}} \quad \text{and} \quad \nabla \times \mathbf{F} \Rightarrow (i\mathbf{k}) \times \tilde{\mathbf{F}}.$$

The transformed fields are often called plane wave solutions of Maxwell's equations. These fields have infinite spatial extent and so are not physically realizable but many sources can be treated approximately as plane waves, particularly monochromatic (single frequency or small bandwidth) sources. Indeed, many treatments of electromagnetic phenomena will simply posit that there are solutions that have the spatial dependence

FIGURE 2.5. For plane waves, the direction of propagation of the electromagnetic wave is defined by the wave vector $\mathbf{k}$. This direction is orthogonal to both the electric $\tilde{\mathbf{E}}$ and magnetic $\tilde{\mathbf{H}}$ fields.



$\exp(i\mathbf{k} \cdot \mathbf{r})$ and proceed from there. In fact, these treatments are skipping over the derivation from the Fourier transform. While the exponential terms are technically infinite, we anticipate that real fields will have finite support and the integrals will be finite due to the properties of the transformed field $\tilde{\mathbf{E}}(\mathbf{k})$. We will come back to this point subsequently.

Even with the simplifications associated with integral transforms, we are still left with a series of vector equations. One possible approach to dealing with Maxwell's equations utilizes an alternative formulation in terms of potentials. One might ask the question, as regards equation 2.12, "What sort of function has no divergence?" It is possible to construct one by simply taking the curl of *any* vector function. Similarly, one can use the fact that the curl of the gradient of any scalar function also vanishes. In the static case ($\omega = 0$), we can use this result to solve directly for the electric field in equation 2.13.

EXERCISE 2.8. For some vector function $\mathbf{F} = (F_x, F_y, F_z)$ show explicitly that $\nabla \cdot (\nabla \times \mathbf{F}) = 0$. Show also that for a scalar function F that $\nabla \times (\nabla F) = 0$.

More generally, the time-dependent electric field cannot be just the gradient of a scalar function. If we make the decision that we shall define the magnetic induction $\mathbf{B}$ in terms of a vector potential $\mathbf{A}$, then the choice

$$(2.26) \quad \mathbf{B} = \nabla \times \mathbf{A}$$

will automatically solve equation 2.12, for any choice of $\mathbf{A}$. If we also make the choice²

$$(2.27) \quad \mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t},$$

then for any choice of the scalar function V , equation 2.13 will also be solved.

EXERCISE 2.9. Use the definitions of equations 2.26 and 2.27 and demonstrate that the Ampère-Maxwell equation 2.14 leads to a wave equation for the vector potential.

So, with these definitions of the potentials, we have a broad capability of solving Maxwell's equations, at least in source-free regions. We can look for functions V and $\mathbf{A}$ that satisfy boundary conditions for any particular problem and construct the electric and magnetic fields accordingly. We note, though, that equations 2.26 and 2.27 admit some ambiguity in the definitions of the potentials $\mathbf{A}$ and V . If these potentials provide solutions $\mathbf{E}$ and $\mathbf{B}$ to Maxwell's equations, then so do the modified potentials

$$(2.28) \quad \mathbf{A}' = \mathbf{A} + \nabla\phi \quad \text{and} \quad V' = V - \frac{\partial\phi}{\partial t},$$

where ϕ is an arbitrary scalar field $\phi = \phi(t, \mathbf{r})$. This transformation is known as a **gauge** transformation³ and represents a non-obvious symmetry of Maxwell's equations. Modern theories of elementary particles are all gauge field theories; we shall discuss this property in more detail subsequently.

EXERCISE 2.10. Show that the modified potentials $\mathbf{A}'$ and V' produce the same electric and magnetic fields as the potentials $\mathbf{A}$ and V .

What we would like to do next is study how to devise solutions that involve sources. In this case, the potentials provide a means for doing just that. The mathematics is rather involved, so we will skip over the bulk of the derivations. A general strategy for dealing with source terms involves computing the Green's function. Suppose that we have some differential equation involving the function $F(x)$. We can write this generically as follows:

$$\mathcal{D}F(x) = s(x),$$

²We use the symbol V for the electric potential, as is traditional and distinguish it from the volume $\mathcal{V}$ by using a different font. Students will have to be constantly vigilant, as such subtleties can be obscured when copying equations from the chalkboard.

³The term *gauge* is due to Hermann Weyl, who was studying the effects of scaling transformations on Lagrangian systems.

where $\mathcal{D}$ represents all of the derivative operators and other functions multiplying F and s is the source term. The Green's function is defined as the function that satisfies the differential equation with a delta function source:

$$\mathcal{D}G(x, x_1) = \delta(x - x_1).$$

The function F now can be recovered by performing the following integral⁴:

$$(2.29) \quad F(x) = \int dx_1 G(x, x_1) s(x_1).$$

Now one might question how one goes about solving an equation where the source term is singular but, recall, we have already derived an integral representation for the delta function. If we work in the transform domain, things are not as bleak as it might seem at first.

In any case, the potentials for Maxwell's equations can be solved using this strategy. We find that the potentials have a surprisingly simple form:

$$(2.30) \quad V(t, \mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d^3\mathbf{r}_1 \frac{\rho(t_1, \mathbf{r}_1)}{|\mathbf{r} - \mathbf{r}_1|}$$

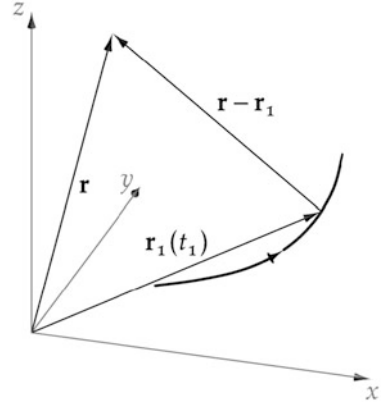
$$(2.31) \quad \mathbf{A}(t, \mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3\mathbf{r}_1 \frac{\mathbf{J}(t_1, \mathbf{r}_1)}{|\mathbf{r} - \mathbf{r}_1|},$$

where $t_1 = t \pm |\mathbf{r} - \mathbf{r}_1|/c$ is the time at the source location. There are two possible solutions, known as the advanced (+) and retarded (−) solutions. We think of the retarded solutions as being the physical ones and shall subsequently ignore the advanced solutions. Here we are invoking a causality argument: the fields at some distance $d = |\mathbf{r} - \mathbf{r}_1|$ from the source cannot know about changes in the source until the time that it would take light to travel that distance. Using the advanced solution would give us knowledge of the source motion prior to its actual motion.

We have utilized a fair amount of complex analysis to this point, by which we mean complex numbers not just complicated formulas. It will take time before students can become accustomed to talking about imaginary numbers with a straight face. It is an equally disheartening experience to try to explain to one's colleagues that we are dealing with retarded solutions; there is inevitable gleeful commentary that all of physics is retarded and that all involved in the study of such an inane subject must have some form of mental handicap. One should display forbearance.

⁴We're skipping over some details of other terms that generally can be forced to vanish by assuming that the Green's function and the source term vanish sufficiently fast at large distances.

FIGURE 2.6. A charged particle follows the path $\mathbf{r}_1(t_1)$ as indicated by the dark curve. The fields are measured at some distant point $\mathbf{r}$.



2.3. Point sources

The results presented in equations 2.30 and 2.31 are deceptively simple. The integrands are actually singular at the source point $\mathbf{r} = \mathbf{r}_1$ and, thus, actually performing the integrals requires some mathematical finesse. Even for the case of a single point source, where $\rho = Q\delta^3(\mathbf{r} - \mathbf{r}_1)$, it is not as simple as just collapsing the integrals, owing to the fact that we have to take into consideration the retarded time. Nevertheless, these intricacies can be overcome, with the result that the potentials for a moving point source can be written as follows:

$$(2.32) \quad V(t, \mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{|\mathbf{r} - \mathbf{r}_1| - (\mathbf{r} - \mathbf{r}_1) \cdot \mathbf{v}_1/c} \quad \text{and}$$

$$(2.33) \quad \mathbf{A}(t, \mathbf{r}) = \frac{\mu_0}{4\pi} \frac{q\mathbf{v}_1/c}{|\mathbf{r} - \mathbf{r}_1| - (\mathbf{r} - \mathbf{r}_1) \cdot \mathbf{v}_1/c},$$

where q is the charge and $\mathbf{v}_1 = d\mathbf{r}_1/dt_1$ is the velocity of the point charge at the retarded time. These are known as the Liénhard-Wiechert potentials.

EXERCISE 2.11. A complicating factor in the analysis of the Liénhard-Wiechert potentials is that the time t_1 at the source point depends on the position of the field point $\mathbf{r}$. Consequently, derivatives of t_1 with respect to the spatial components of $\mathbf{r}$ do not vanish. We have

$$t_1 = t - \frac{1}{c} \sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}.$$

What is the derivative with respect to x :

$$\frac{\partial t_1}{\partial x} = ?$$

What then is the result of applying the gradient operator ∇t_1 ?

We can now recover the fields through equations 2.27 and 2.26. The algebra is tedious but manageable. We obtain the following:

(2.34)

$$\mathbf{E}(t, \mathbf{r}) = \frac{q}{4\pi\epsilon_0} \frac{|\mathbf{r} - \mathbf{r}_1|}{[|\mathbf{r} - \mathbf{r}_1| - (\mathbf{r} - \mathbf{r}_1) \cdot \mathbf{v}_1/c]^3} \left\{ \left(1 - \frac{v_1^2}{c^2} \right) \left(\frac{\mathbf{r} - \mathbf{r}_1}{|\mathbf{r} - \mathbf{r}_1|} - \frac{\mathbf{v}_1}{c} \right) + (\mathbf{r} - \mathbf{r}_1) \times \left[\left(\frac{\mathbf{r} - \mathbf{r}_1}{|\mathbf{r} - \mathbf{r}_1|} - \frac{\mathbf{v}_1}{c} \right) \times \frac{\mathbf{a}_1}{c^2} \right] \right\}$$

(2.35)

$$\mathbf{B}(t, \mathbf{r}) = \frac{\mathbf{r} - \mathbf{r}_1}{c|\mathbf{r} - \mathbf{r}_1|} \times \mathbf{E}(t, \mathbf{r}).$$

Here, $\mathbf{a}_1 = d^2\mathbf{r}_1/dt_1^2$ is the acceleration of the charge.

EXERCISE 2.12. As a sanity check, set $\mathbf{v}_1 = 0$ and $\mathbf{a}_1 = 0$ in equation 2.34 and see if you recover Coulomb's law.

These results are reasonably gruesome but we can make some immediate observations. First, as we can see from figure 2.6, the vector $\mathbf{r} - \mathbf{r}_1$ provides the direction from the point charge to the field point at the retarded time. Second, from equation 2.33, we see that the magnetic field is perpendicular to both that direction and the electric field. Recall that the plane wave solutions we discussed previously also had the magnetic field perpendicular to the electric field and the direction of propagation.

In deriving complicated results like those found in equations 2.34 and 2.35, it is a very good strategy to conduct dimensional analysis throughout the derivation, to avoid unfortunate algebraic mistakes. Upon completion, it is also a good idea to check the dimensionality of your result. Dimensionality is a bit difficult to assess in electromagnetics but we can recall from Coulomb's law that the electric field of a point charge has a factor of $(4\pi\epsilon_0)^{-1}$ multiplying the charge divided by distance squared. We see just such a term in the first part of equation 2.34. This means that the remaining factor in the curly brackets must be dimensionless. In fact we have organized the result in a fashion that makes this reasonably apparent.

EXERCISE 2.13. Express equation 2.34 dimensionally and verify that the terms in the curly brackets are dimensionless and that the remaining term involving $\mathbf{r}$ scales like (L^{-2}) .

Moreover, we can observe that the first term in the curly braces has no dependence on the distance $d = |\mathbf{r} - \mathbf{r}_1|$, only on the direction from the source. As a result, this first term will have an overall inverse square dependence on d . It is, thus, the modification of the Coulomb force law that incorporates a moving (constant velocity) charge. The second term in the curly braces is proportional to the acceleration of the charge and

contains an additional factor of d , so that asymptotically the field will scale inversely with d . This component is termed the radiation component of the field.

To understand what we mean by the term radiation, we should say that the Poynting vector $\mathbf{S}$ defines the energy flux of the electromagnetic field. We'll go into a bit more detail subsequently but the definition of the Poynting vector is as follows:

$$(2.36) \quad \mathbf{S} = \mathbf{E} \times \mathbf{H}.$$

Energy conservation of the electromagnetic fields will be shown shortly to take the following form:

$$(2.37) \quad \frac{d\mathcal{E}}{dt} = \frac{d}{dt} \int_{\mathcal{V}} d^3\mathbf{r} \frac{1}{2} [\epsilon_0 E^2 + \mu_0 H^2] + \int_{\partial\mathcal{V}} d^2\mathbf{r}_1 \mathbf{n}(\mathbf{r}_1) \cdot \mathbf{S},$$

where $\mathcal{E}$ is the total energy within the volume $\mathcal{V}$ and the last term in the equation represents the energy flux through the boundary $\partial\mathcal{V}$. Radiation will be defined as the fields that propagate to infinity, that is only those components that remain finite as the bounding surface $\partial\mathcal{V}$ is removed to infinity.

If we take the bounding surface to be a large sphere, then the differential elements for the surface integral will be

$$d^2\mathbf{r}_1 = r_1^2 \sin\theta \, d\theta \, d\varphi,$$

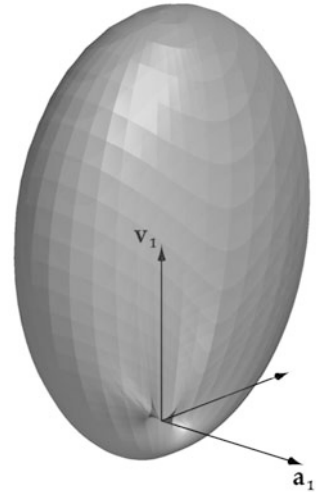
which clearly scales like the distance squared. If we consider actually computing the product $\mathbf{E} \times \mathbf{H}$ from equations 2.34 and 2.35, we should note that we will obtain, from the product of the first term in the curly braces with itself, a term that scales like d^{-4} . Overall, this will scale like d^{-2} (due to the d^2 coming from the differential element) and will vanish as $d \rightarrow \infty$. We will also obtain terms, from the product of the first term in the curly braces with the second term, that scale like d^{-3} and, overall, like d^{-1} and will likewise vanish. The only terms that can possibly remain will be the one obtained from product of the second term in the curly braces with itself, which is independent of d and, thus, can remain finite as $d \rightarrow \infty$.

EXERCISE 2.14. Examine the terms in the Poynting vector dimensionally and verify the assertions made in the discussion above.

We should note that our somewhat haphazard analysis of the previous paragraph can be made rigorous, so the mathematically aware need not be disheartened. Nevertheless, electromagnetics is the discipline where all of the equations become difficult. Rather than mindlessly trying to perform all of the integrals one encounters, it is generally better to try to understand something about the integrals before trying to solve them.

In this particular instance, we needn't even perform the bulk of the integrations as we are only seeking solutions that will remain finite at large distances.

FIGURE 2.7. When the acceleration is perpendicular to the direction of motion, the radiation is predominantly in the forward direction. Here, the velocity is $0.4c$.



Using these results for the fields generated by an accelerated charge and after another long algebraic effort, we can arrive at the following result for the radiation field of a charged particle that is being accelerated perpendicular to its velocity:

$$(2.38) \quad |\mathbf{S}_{\text{rad}}| = \frac{\mu_0 q^2 a_1^2}{16\pi^2 c} \frac{[1 - (v_1/c)^2 \cos^2 \theta]^2 - [1 - (v_1/c)^2] \sin^2 \theta \cos^2 \varphi}{[1 - (v_1/c) \cos \theta]^5},$$

where here a_1 is the magnitude of the acceleration, θ measures the angle between the field point and the velocity $\mathbf{v}_1$ and φ is the angle between the field point and the acceleration $\mathbf{a}_1$. The energy flux is illustrated in figure 2.7.

This radiation is known as synchrotron radiation and is a serious problem that arises when constructing particle accelerators, particularly for electron beams. If we utilize magnetic fields to bend the electron beam into a circle, a substantial portion of the energy injected into the electrons will be radiated in the form of x-rays. From figure 2.7, we see that the radiation is emitted in the direction of propagation, perpendicular to the (centripetal) acceleration. Initially considered a parasitic problem, there are now synchrotrons built explicitly for the purpose of generating intense x-ray beams for crystallography. In these machines, no one is doing physics with the electron beams; they are, instead, studying the three-dimensional structures of protein crystals and other novel materials.

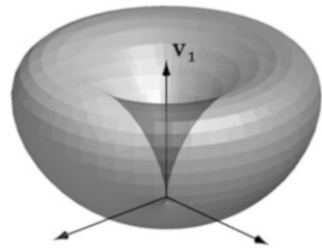
EXERCISE 2.15. Use the `Manipulate` function to study the behavior of the radiation flux from equation 2.38 as a function of the velocity (v_1/c). Use the `SphericalPlot3D` function to plot $\mathbf{S}_{\text{rad}}$.

For the other special case of an acceleration that is collinear with the velocity, we obtain the following result:

$$(2.39) \quad |\mathbf{S}_{\text{rad}}| = \frac{\mu_0 q^2 a_1^2}{16\pi^2 c} \frac{\sin^2 \theta}{[1 - (v_1/c) \cos \theta]^5}.$$

The radiation in this case, depicted in figure 2.8, is symmetrical around the direction of travel (no φ dependence) and is zero at $\theta = 0$; it is known as *bremsstrahlung*. As the velocity increases, the angle of maximum flux continues to decrease, such that at extreme relativistic velocities, the maximum flux is directed at an angle that approaches zero.

FIGURE 2.8. When the velocity and acceleration are collinear, the radiation pattern is symmetrical about the direction of motion and is increasingly tilted into the forward direction as the velocity increases. Here the velocity is $0.4c$.



EXERCISE 2.16. Use the `Manipulate` function to study the behavior of the radiation flux from equation 2.39 as a function of the velocity (v/c). Use the `SphericalPlot3D` function to plot $\mathbf{S}_{\text{rad}}$. What happens as $v/c \rightarrow 1$?

2.4. Relativistic Formulation

We've skipped over a number of details to get to this point but it is time to confront some crucial issues directly. For example, we can ask what happens when we apply a Lorentz transformation to Maxwell's equations and the answer is rather disappointing: the electric and magnetic fields transform in a very complicated manner. Assigning students the task of deriving those transformations is commonplace in introductory texts but not terribly illuminating. We know that the fields are solutions to a wave equation and that the wave equation is invariant under a Lorentz transformation. So, let us instead make use of Einstein's adventures into the mathematical hinterlands and ask an alternative question: Is there some way to write Maxwell's equations that makes the question of Lorentz invariance obvious and straightforward.

As Einstein discovered, the answer is yes. Yes, it is possible to do so but it took Einstein a decade to fully comprehend the required mathematical infrastructure. In Einstein's defense, the mathematics was new and not yet widely understood by anyone outside the mathematical community. So, Einstein had to translate mathematics into physics, learning new vocabulary and constructing useful examples. Mathematicians are concise writers and often satisfied with existence and uniqueness proofs. Constructive proofs and nontrivial examples are often in short supply.

In any event, the appropriate language is provided by tensor calculus, where tensors are the mathematical extension of vectors. Unfortunately, the additional benefits of tensor calculus comes with a significant amount of mathematical baggage that can be perplexing to physicists. We shall endeavor to persevere. We are interested in using tensor calculus, which is the extension of calculus to multidimensional objects and non-Cartesian geometries. We will find the notation to be nearly impenetrable and the motivation completely opaque but, if we hold fast to a few touchstones, we will emerge with at least a modest appreciation for the underlying mathematical structure.

If we look at the definition of some vector $\mathbf{v}$ in a space of N dimensions, we usually begin by defining a basis $\mathbf{e}_1, \dots, \mathbf{e}_N$, where the $\mathbf{e}_i$ are linearly independent. Actually, impressive works in modern geometry demonstrate that we do not need to define a basis at all; everything can be done in a fashion that is basis-independent. For the moment, we shall proceed in a more or less straightforward fashion, one that should be reasonably familiar. In previous examples studied in physics class, we would normally construct the $\mathbf{e}_i$ to be orthonormal but this is not a requirement and, in some cases, is not desirable. For example, in dealing with crystals, we would want to choose a basis that aligns with the crystal axes. In other cases, using a curvilinear coordinate system may prove to be advantageous.

With respect to the basis formed by the $\mathbf{e}_i$, the vector $\mathbf{v}$ can be resolved into components:

$$\mathbf{v} = \sum_{i=1}^N v_i \mathbf{e}_i = (v_1, \dots, v_N).$$

In order to determine the values of the components v_i , we need to use the dual vectors $\tilde{\mathbf{e}}_i$. In crystallography (and in three dimensions), these are known as the reciprocal vectors and can be constructed as follows:

$$(2.40) \quad \tilde{\mathbf{e}}_1 = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{\mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3)}, \quad \tilde{\mathbf{e}}_2 = \frac{\mathbf{e}_3 \times \mathbf{e}_1}{\mathbf{e}_2 \cdot (\mathbf{e}_3 \times \mathbf{e}_1)}, \quad \text{and} \quad \tilde{\mathbf{e}}_3 = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{\mathbf{e}_3 \cdot (\mathbf{e}_1 \times \mathbf{e}_2)}.$$

These dual vectors have the following property:

$$(2.41) \quad \tilde{\mathbf{e}}_i \cdot \mathbf{e}_j = \delta_{ij},$$

where the δ function here is zero if $i \neq j$ and one if $i = j$. Note that, in Cartesian coordinates, the dual vectors are equivalent to the original basis vectors but this is not generally true.

EXERCISE 2.17. Use the definitions from equation 2.40 and show that equation 2.41 is true.

Suppose now that we define a different set of basis elements $\mathbf{e}'_i$, where we have scaled the original $\mathbf{e}_i$ by some factor. This would occur, for example, if we switch between using meters and centimeters as our units of measure. What happens then to the values of the vector components? If our original coordinate values x_i were provided in terms of meters, then the new coordinate values x'_i will be one hundred times larger, reflecting the change to units of centimeters. Mathematicians have termed this property **contravariant**. That is, the values of the components v_i scale inversely to the change of scale of the basis vectors.

Consider now transforming a dual vector

$$\tilde{\mathbf{v}} = \sum_{i=1}^N \tilde{v}_i \tilde{\mathbf{e}}_i.$$

If each of the basis elements is scaled by a factor, then the dual basis elements will scale inversely to that factor. The components of the dual vector will scale proportionally. That is, if our units change from meters to centimeters, the components $\tilde{v}'_i$ of the dual vector will also be one hundred times smaller than the components $\tilde{v}_i$ of the original vector. This property is termed **covariant**.

EXERCISE 2.18. Consider the following basis vectors:

$$\mathbf{e}_1 = (1, 0, 0), \quad \mathbf{e}_2 = (0.2, 1, 0) \quad \text{and} \quad \mathbf{e}_3 = (0, 1.3, 1.8).$$

What are the reciprocal vectors $\tilde{\mathbf{e}}_i$? What are the components of the vector $\mathbf{v} = 3\hat{\mathbf{x}} + 4\hat{\mathbf{y}} + 2.5\hat{\mathbf{z}}$ in the $\mathbf{e}_i$ basis? What are the components of $\mathbf{v}$ in the reciprocal basis?

Repeat the analysis for the new basis $\mathbf{e}'_i = \mathbf{e}_i/100$.

In moving forward, we face a serious notational problem. We have heretofore utilized a bold typeface to identify vectors but we don't have a convenient typographical mechanism for identifying multidimensional tensors. In many matrix algebra classes, one will find that upper-case variables will

be restricted to matrices and lower-case variables will be vectors. Consequently, the following equation:

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

can readily be interpreted to mean the multiplication of a matrix and a vector that returns a vector. Unfortunately, we have already used $\mathbf{E}$ to mean the vector electric field and $\mathbf{A}$ to mean the magnetic potential, so it is a bit late to enforce the case-sensitive strategy.

Worse, tensors are not restricted to two-dimensional entities, so we do not have a convenient typographical means of distinguishing one-, two-, three- and four-dimensional objects by their typeface. The most common notational convention used in physics explicitly expresses the equations in component form, using superscript indices to denote contravariant components and subscript components to denote covariant components. The number of indices denotes the dimension or rank of the tensor. For example, R^j_{klm} denotes a fourth-rank tensor with one contravariant index and three covariant indices. Our matrix equation now reads as follows:

$$\sum_j A_{ij} x_j = b_i.$$

This strategy does not lead to particularly elegant representations. Einstein became so distressed over repeating all of the summation symbols in his work that he invented a new convention: repeated indices are implicitly summed:

$$A^{ij} x_j = b^i.$$

All three of these equations are intended to mean the same thing.

Mathematicians have developed somewhat more elegant forms but are generally much more abstract, making it difficult for non-experts to follow. For this first time through the material, we shall hew to the physics practice, using the component notation but we shall forego the use of the Einstein summation convention. We'll leave it to the students to clean up the formulas as we proceed. Note that a particularly unwieldy result of this notational practice is that by the symbol v^i , we can mean both the i th component of a contravariant vector $\mathbf{v}$ or the vector itself. Alas, one will often have to infer the meaning from context but we shall attempt to clarify as necessary.

Let us briefly define mathematically what we mean by covariant and contravariant. If we have two separate coordinate systems $\mathbf{x} = (x^1, \dots, x^N)$ and $\mathbf{y} = (y^1, \dots, y^N)$ then the contravariant vector whose components are v^i will

transform as follows:

$$(2.42) \quad v^i(\mathbf{y}) = \sum_{j=1}^N \frac{\partial y^i}{\partial x^j} v^j(\mathbf{x}).$$

Similarly, the covariant vector whose components are denoted by v_i will transform as follows:

$$(2.43) \quad v_i(\mathbf{y}) = \sum_{j=1}^N \frac{\partial x^j}{\partial y^i} v_j(\mathbf{x}).$$

It is a bit difficult to see amidst all the indices but there is an inverse relationship between the coefficients:

$$\frac{\partial y^i}{\partial x^j} = \left[\frac{\partial x^j}{\partial y^i} \right]^{-1}.$$

We won't expand further on this at the moment but it does formally define what we mean by tensors. We should note also that we have used a notation in equations 2.42 and 2.43 where we think of the transformed vector as an explicit function of the new coordinates $\mathbf{y}$, where the original vector was a function of the coordinates $\mathbf{x}$. This strategy is not standard, often the transformed vector is $\mathbf{x}'$, but saves us from further decorating the symbol v with a prime or tilde to denote that a transformation has occurred.

Let us now revisit the Maxwell equations 2.11–2.14 and see what happens if we recast them into tensor form. First, we note that we must work in four-dimensional spacetime and will choose an invariant interval $ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$. This choice makes timelike vectors positive and spacelike vectors negative but students should beware that this choice is by no means unanimous. Other authors will work with the signs reversed. This will lead to differences in intermediate results but the same answers eventually. There are also differences as to what to call the terms. Many authors consider $c dt = x^0$, the zeroth element of the position vector, which has spatial components x^1 , x^2 and x^3 . Others will place time into the fourth position $c dt = x^4$. As the *Mathematica* software utilizes indices that run from 1 to N , we'll just call $c dt$ the first element of the position vector and the x -value will occupy the second.

The Maxwell equations represent first order differential equations for the fields. The obvious choice for a four-dimensional derivative operator is the following:

$$(2.44) \quad \frac{\partial}{\partial x^i} = \left\{ \frac{\partial}{c \partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\}.$$

EXERCISE 2.19. A more compact notation defines the differential operator ∂_i as

$$\partial_i \equiv \frac{\partial}{\partial x^i}.$$

Convince yourself that ∂_i is a covariant vector, hence the lower index.⁵ Hint: refer to the definitions in equation 2.42 and 2.43.

If we now revisit equations 2.27 and 2.26, we would like to define a four-potential from V and $\mathbf{A}$. From equation 2.27, we know that the dimension of the gradient of the potential V must be $(V \cdot m^{-1})$. Furthermore, we can infer that the vector potential $\mathbf{A}$ must have dimension of $(V \cdot s \cdot m^{-1})$. As a result, we can define a four-potential A that has consistent dimensions as follows:

$$(2.45) \quad A^i = (V/c, A_x, A_y, A_z) = (V/c, \mathbf{A}).$$

The covariant potential will be given by the following:

$$(2.46) \quad A_i = (V/c, -A_x, -A_y, -A_z) = (V/c, -\mathbf{A}).$$

What happens if we now apply the differential operator to the four potential? We find, in fact, the following:

$$(2.47) \quad \frac{\partial A_k}{\partial x^i} = \begin{bmatrix} \frac{\partial V/c}{c \partial t} & -\frac{\partial A_x}{c \partial t} & -\frac{\partial A_y}{c \partial t} & -\frac{\partial A_z}{c \partial t} \\ \frac{\partial V/c}{\partial x} & \frac{\partial A_x}{\partial x} & \frac{\partial A_y}{\partial x} & \frac{\partial A_z}{\partial x} \\ \frac{\partial V/c}{\partial y} & \frac{\partial A_x}{\partial y} & \frac{\partial A_y}{\partial y} & \frac{\partial A_z}{\partial y} \\ \frac{\partial V/c}{\partial z} & \frac{\partial A_x}{\partial z} & \frac{\partial A_y}{\partial z} & \frac{\partial A_z}{\partial z} \end{bmatrix}.$$

Now, let us subtract the transpose of this object:

$$(2.48) \quad \frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k} = \begin{bmatrix} 0 & -\frac{\partial V/c}{\partial x} - \frac{\partial A_x}{c \partial t} & -\frac{\partial V/c}{\partial y} - \frac{\partial A_y}{c \partial t} & -\frac{\partial V/c}{\partial z} - \frac{\partial A_z}{c \partial t} \\ \frac{\partial V/c}{c \partial t} + \frac{\partial A_x}{\partial x} & 0 & \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} & \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \\ \frac{\partial V/c}{c \partial t} + \frac{\partial A_y}{\partial y} & \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} & 0 & \frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y} \\ \frac{\partial V/c}{c \partial t} + \frac{\partial A_z}{\partial z} & \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} & \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} & 0 \end{bmatrix}.$$

⁵The operator is also known to mathematicians as a 1-form. There is a veritable ocean of mathematical literature on the extension of simple, ordered tuples (vectors) into more complex entities (tensors).

We shall define this last object to be the electromagnetic field tensor F . We can recognize the components of the tensor:

$$(2.49) \quad F_{ik} \equiv \frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{bmatrix}.$$

EXERCISE 2.20. Write out the components of equations 2.26 and 2.27 and convince yourself that the identifications made in equation 2.49 are correct.

We haven't yet shown that F is actually a tensor but it is not too difficult to do so. We want now to take the four-divergence of the field tensor but, to do so, we must apply one of the rules associated with tensors. To form the equivalent of a dot product, we utilize the metric tensor g :

$$(2.50) \quad \sum_{i=1}^4 \sum_{k=1}^4 g_{ik} a^i b^k = \sum_{i=1}^4 a^i b_i = \sum_{k=1}^4 a_k b^k.$$

Here, we note in the last two terms that the transformation of a vector by the metric tensor converts it from a contravariant to covariant vector, lowering the index. We can also define the conjugate metric tensor g^{ik} that is the inverse of g_{ik} . We can show

$$\sum_{j=1}^N g_{ij} g^{kj} = \delta_i^k = \delta_{ik},$$

where here the delta function is one when $i = k$ and zero otherwise.

EXERCISE 2.21. Formally, raising and lowering indices can be accomplished by utilizing the metric tensor g , where here we are using the form:

$$g_{ij} = g^{ij} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

If we define the contravariant vector $x^i = (ct, x, y, z)$, what is the covariant vector x_i ? We defined $A^i = (V/c, A_x, A_y, A_z)$. What is the covariant vector A_i ?

The four-divergence of the field tensor is computed as follows:

$$\begin{aligned}
 \sum_{i=1}^4 \sum_{l=1}^4 g^{il} \frac{\partial}{\partial x^l} F_{ik} &= \left[\frac{\partial}{c \partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z} \right] \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{bmatrix} \\
 (2.51) \quad &= \begin{bmatrix} \frac{\partial E_x}{c \partial x} + \frac{\partial E_y}{c \partial y} + \frac{\partial E_z}{c \partial z} \\ \frac{c^2 \partial t}{\partial E_x} - \frac{\partial y}{\partial B_z} + \frac{\partial z}{\partial B_y} \\ \frac{c^2 \partial t}{\partial E_y} - \frac{\partial z}{\partial B_x} + \frac{\partial x}{\partial B_z} \\ \frac{c^2 \partial t}{\partial E_z} - \frac{\partial x}{\partial B_y} + \frac{\partial y}{\partial B_x} \end{bmatrix}
 \end{aligned}$$

If we now recall that $\epsilon_0 \mu_0 = 1/c^2$, we can recognize these results as just the component terms of the Maxwell equations 2.11 and 2.14. We are lacking the source terms, though. If we define the four-current density J as follows:

$$(2.52) \quad J^i = (c\rho, J_x, J_y, J_z),$$

then we can write Maxwell's equations in the following form:

$$(2.53) \quad \sum_{i=1}^4 \frac{\partial}{\partial x_i} F_{ij} = \mu_0 J_j \quad \text{and} \quad \sum_{i=1}^4 \frac{\partial}{\partial x_i} G_{ij} = 0,$$

where we have made use of the dual electromagnetic tensor G :

$$(2.54) \quad G_{ik} = \begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z/c & -E_y/c \\ -B_y & -E_z/c & 0 & E_x/c \\ -B_z & E_y/c & -E_x/c & 0 \end{bmatrix}.$$

EXERCISE 2.22. Check the dimensionality of the components of the four-current vector J^i .

EXERCISE 2.23. Expand Maxwell's equations 2.11–2.14 into components and compare to the terms of equation 2.53. Convince yourself that they are just different representations of the same terms.

EXERCISE 2.24. A more common expression of Maxwell's equations is given as follows:

$$\partial_i F^{ik} = \mu_0 J^k \quad \text{and} \quad \partial_i G^{ik} = 0,$$

where we have used the Einstein summation convention and the contravariant form of the field tensor and four-current. Write out

the components and show that this results in the same equations (up to an overall sign) as those in equation 2.53.

The purpose of this exercise has been to put Maxwell's equations into tensor form because it is in this form that Lorentz transformations have a simple expression. The Lorentz transformations are just a special set of coordinate transformations that consist of the rotation matrices that mix the spatial components of a four-vector or four-tensor and boost matrices that mix the time and space coordinates. In this representation, the Maxwell equations 2.53 are manifestly invariant under Lorentz transformation. The components F_{ij} will be altered but the form of the equation does not change.

EXERCISE 2.25. Consider a Lorentz boost along the x -axis. This is described by the following matrix:

$$B_{ik} = \begin{bmatrix} \cosh \zeta & \sinh \zeta & 0 & 0 \\ \sinh \zeta & \cosh \zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

What is the Lorentz transform of the four-current density J_i ? What is the Lorentz transformation of the field tensor F_{ik} ?

EXERCISE 2.26. There are two scalar invariants that can be constructed from the field tensor F and its dual G . What are the values of the following expressions:

$$\sum_{i=1}^4 \sum_{k=1}^4 F^{ik} F_{ik} \quad \text{and} \quad \sum_{i=1}^4 \sum_{k=1}^4 F^{ik} G_{ik}?$$

2.5. Solitons

We have seen that Maxwell's equations provide an accurate representation of electromagnetic phenomena, making numerous predictions that have been repeatedly verified experimentally. Moreover, when cast into tensor form, the equations are consistent with Einstein's theory of special relativity. The problem that we face in trying to develop a classical model for the photon is that the properties of a photon are largely unspecified. This is due primarily to the fact that photons classically do not interact with one another. Hence, we cannot scatter a photon from another photon and learn anything about its structure. We can speculate that photons are somehow pulses of electromagnetic energy that have finite spatial extent.

The notion of a finite blob of electromagnetic energy arose initially when the German physicist Max Planck considered the problem of blackbody radiation from a statistical point of view. If one can construct the partition function $\mathcal{Z}$, which is quite literally the sum over all possible states, then one can determine all of the interesting thermodynamic properties of a system. Planck provided the *ansatz* that electromagnetic energy could be dealt with in a similar fashion as was used to enumerate the distribution of particles in a box: that is, the distribution of electromagnetic energy within the box was given by the following formula:

$$(2.55) \quad \mathcal{Z} = \sum_{n=0}^{\infty} e^{-nh\nu/kT} = \frac{1}{1 - e^{-h\nu/kT}},$$

where $h\nu$ is the energy of the photons, T is the absolute temperature and k is a constant, now known as Boltzmann's constant. From this result, we can obtain the average number of photons $\bar{n}$ by noticing that

$$\frac{\partial \mathcal{Z}}{\partial h\nu} = -\frac{n}{kT}.$$

From this, we obtain Planck's result:

$$(2.56) \quad \bar{n} = \frac{1}{e^{h\nu/kT} - 1}.$$

Planck's suggestion turns out to solve a conundrum that arose when earlier attempts at trying to explain the behavior of blackbody radiation tried to integrate over the electromagnetic energy in the box. This strategy results in a divergent integral, whereas Planck's summation was finite.

EXERCISE 2.27. Plot the function $f(x) = 1/(\exp(x) - 1)$.

So, the introduction of the photon into physics literature was really a numerical device that succeeded because it (subtly) introduced a frequency cutoff into the partition function. This is not particularly obvious but was later appreciated by Einstein in his *annus mirabilis* paper on the subject, where he introduced the nomenclature *Lichtquant* to describe the quantized portion of electromagnetic energy.

If asked to describe a photon, the image that undoubtedly springs to mind is a small glowing ball that travels at the speed of light. That is often how photons are depicted but can we be more quantitative about this picture? Can we find solutions of Maxwell's equations that are spatially confined for all time? Some years ago, the physicist James Brittingham conducted a search for just this, subject to the requirements that solutions

- (1) satisfy the homogeneous Maxwell's equations,
- (2) have a three-dimensional pulse structure,
- (3) have no charge,

- (4) move at light velocity in straight lines,
- (5) are nondispersive and
- (6) have finite energy.

If we assume propagation in the z -direction, there are well-known plane wave solutions that are functions of $z + ct$ and $z - ct$ that represent left- and right-propagating solutions, respectively. Brittingham looked for solutions that were, instead, functions of the product $f_1(z + ct)f_2(z - ct)$. He discovered that the following function is a solution of the homogeneous wave equation:

$$(2.57) \quad \Psi(t, \zeta, z) = \frac{1}{4\pi i} \int_{-\infty}^{\infty} dq w(q) e^{iq(z+ct)} \frac{e^{-q\zeta^2/\eta}}{\eta},$$

where $\eta = z_0 + i(z - ct)$ and $w(q)$ is an arbitrary weighting function. An example of such a solution is depicted in figure 2.9.

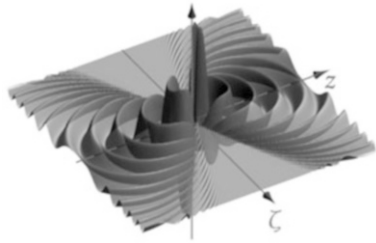


FIGURE 2.9. The simple, rotationally symmetric focus wave mode has a Gaussian profile in the transverse ζ -direction and falls off exponentially in the z -direction.

EXERCISE 2.28. Define the integrand of the function $\Psi(t, \zeta, z)$ in *Mathematica*. (Ignore the weighting function w .) Use the `D` function to show that Ψ is a solution to the wave equation in cylindrical coordinates:

$$\left[\frac{\partial^2}{\partial \zeta^2} + \frac{1}{\zeta} \frac{\partial}{\partial \zeta} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \Psi(t, \zeta, z) = 0.$$

EXERCISE 2.29. Plot the function $\Psi(t, \zeta, z)$ over the domain $-5 \leq \zeta \leq 5$ and $-5 \leq z \leq 5$ for $q = 2\pi$ and $z_0 = 0.1$. How does the function change as t increases from 0 to 1 in steps of 0.1?

Brittingham termed the solutions focus wave modes and it has subsequently been demonstrated that these satisfy all of the original conditions except finite energy. The focus wave modes are infinite-energy solutions to Maxwell's equations. So too, are the plane wave solutions that we have discussed previously. We believe that real finite-energy solutions can be constructed from superpositions of plane waves, so it may prove that focus wave modes provide a new basis for studying pulse solutions of Maxwell's equations. That is, with suitable weighting of the focus wave

modes or some other, undiscovered solutions, one may yet find a classical description of the photon.

A revisit to figure 1.12, though, calls such a quest starkly into question. Thompson's initial diffraction experiment indicated that single photons produce diffraction patterns that are independent of light intensity. As a result, the photon wave function must have some broader spatial extent than defined by the characteristic wavelength, or else it wouldn't be able to diffract through spatially separated slits. On a larger scale, it is well established that antennas radiate in patterns that can be described as diffraction limited. That is, if the aperture of a radiating body has a characteristic size of d , then the far field radiation will spread with an angle θ that is of the order

$$(2.58) \quad \sin \theta \approx \frac{\lambda}{d},$$

where λ is the wavelength. Depending upon the shape of the aperture and other details, there can be a proportionality factor that can range from one to maybe four or so, but not orders of magnitude.

EXERCISE 2.30. Suppose that a 600 nm photon is emitted from our sun ($d = 7 \times 10^8$ m). What is the diffraction limit angle θ , from equation 2.58 assuming a proportionality factor of 1? At the earth-sun distance ($R_E = 1.5 \times 10^{11}$ m), what is the spatial extent of this photon? At the average Pluto-sun distance ($R_P = 6 \times 10^{12}$ m), what is the spatial extent? The nearest star is approximately 4×10^{16} m distant. When the photon arrives in that solar system (in just over four years), what would be its spatial extent?

Without experimental data to constrain such speculations, physicists have generally not pursued the idea of a classical photon. A number, though, have investigated the general idea of a spatially bounded wavelike entity and have based much of their efforts on the 1834 observation of a localized disturbance in the Union Canal in Scotland. John Scott Russell noted that a barge being pulled through the canal had a mound of water piled in front of it that continued propagating down the canal even after the barge had been stopped. Russell tracked the mound for a couple of miles and reported his findings to the Royal Society and went on to conduct experiments in a wave tank that verified his observations.

Russell's wave of translation, as he called it, can be modelled with a one-dimensional non-linear equation and is the first example of what we now call soliton solutions. Solitons represent spatially bounded solutions of a

non-linear system of equations that serve as models for elementary particles. The equation studied most frequently is the Korteweg-de Vries equation, developed to explain Russell's observations. The equation was initially stated by the French mathematician Joseph Boussinesq in 1871 and subsequently rediscovered by the Dutch mathematicians Diederik Korteweg and his student Gustav de Vries in 1897.⁶

$$(2.59) \quad \frac{\partial \phi(t, x)}{\partial t} + \frac{\partial^3 \phi(t, x)}{\partial x^3} - 6\phi(t, x) \frac{\partial \phi(t, x)}{\partial x} = 0$$

is the canonical form of the equation, where $\phi(t, x)$ represents the wave amplitude. One possible solution is given by the following:

$$(2.60) \quad \phi(t, x) = -\frac{c}{2} \operatorname{sech}^2 \left[\sqrt{c}(x - ct - a)/2 \right],$$

where the wave will propagate to the right with a velocity c and a is an arbitrary constant.

EXERCISE 2.31. Use the **Animate** function to plot the function $\phi(t, x)$. Use $c = 1$ and $a = 0$ and plot ϕ over the domain $-20 \leq x \leq 20$ and for the times $0 \leq t \leq 20$.

EXERCISE 2.32. Suppose that you have two solutions to the Korteweg-de Vries equation,

$$\begin{aligned} \phi_1(t, x) &= -\frac{c_1}{2} \operatorname{sech}^2 \left[\sqrt{c_1}(x - c_1 t)/2 \right] \quad \text{and} \\ \phi_2(t, x) &= -\frac{c_2}{2} \operatorname{sech}^2 \left[\sqrt{c_2}(x - c_2 t)/2 \right]. \end{aligned}$$

Use the **D** function to demonstrate that each satisfies the Korteweg-de Vries equation. Now show that the sum $\phi_1 + \phi_2$ does not.

Equation 2.60 can be found by explicitly trying to construct solutions of the form $\phi(t, x) = \phi(ct - x)$. A general solution for N propagating solitons, each with a different characteristic velocity, was obtained in 1967 by Clifford Gardner, John Greene, Martin Kruskal and Robert Miura who utilized a technique now known as the inverse scattering transform.⁷ Their inverse scattering methodology has been systematized and expanded to a host of other partial differential equations.

⁶Boussinesq published "Théorie de l'ondulation liquide appelée onde solitaire ou de translation, se propageant dans un canal rectangulaire" in the *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences* in 1871. Korteweg and de Vries published "On the change of form of long waves advancing in a rectangular canal and on a new type of long stationary waves" in the *Philosophical Magazine* in 1895.

⁷Gardner *et al.* published "Method for solving the Korteweg-de Vries equation" in the *Physical Review Letters* in 1967.

We can construct the N -soliton solution as follows:

- (1) Define an array P that contains the square roots of the soliton velocities c_i

$$P = \{\sqrt{c_1}, \dots, \sqrt{c_N}\}.$$

- (2) Define the matrix M where the components are given by the following:

$$M_{ik} = \delta_{ik} + \frac{2\sqrt{P_i P_k}}{P_i + P_k} e^{(P_i + P_k)x - (P_i^3 + P_k^3)t - Q_i - Q_k},$$

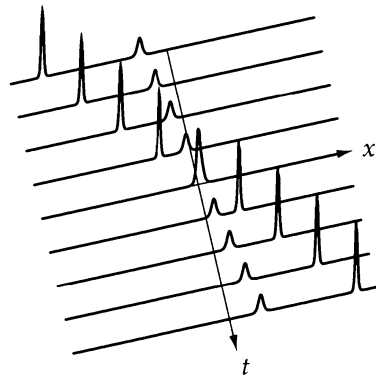
where the Q_i are relative phases of the different waves.

- (3) The solution can be obtained from the following:

$$(2.61) \quad \phi(t, x) = 2 \frac{\partial^2}{\partial x^2} \ln[\det M],$$

where $\det M$ is the determinant of the matrix M .

FIGURE 2.10. Two solitons of different velocities coincide at time zero. Close inspection of the results indicates a phase shift arises due to the interaction.



An example of two solitons, with velocities $c = \{1, 4\}$, is illustrated in figure 2.10. The soliton with the larger velocity also has the larger amplitude. One can see from the figure that the two separate solitons pass through one another, largely unaffected by the interaction. In fact, there is a phase shift associated with the interaction.

EXERCISE 2.33. Use equation 2.61 to compute the two-soliton solution for $P = \{1, 2\}$. Use the **Animate** function to plot the results over the time interval $-10 \leq t \leq 20$ and for the domain $-50 \leq x \leq 50$. Now add the single soliton solutions from equation 2.60 to the plot and adjust their phases so that the initial pulses are aligned. What happens after the interaction at $t = 0$?

The use of the Korteweg-de Vries equation to model simple bounded solutions of a wave equation represents a common practice in physics: use

a (highly) simplified model to test ideas and behaviors before tackling a complex problem. The Korteweg-de Vries equation has non-trivial analytic solutions. If you want to solve the equation numerically, you can test your solver against known solutions. If you are ultimately interested in solving a more complex set of equations, you should have some idea about the robustness of your solver based on your experience with a simpler system.

In studying the problem of constructing solutions to the Korteweg-de Vries equation, Gardner *et al.* found a general strategy for solving nonlinear partial differential equations that, in some sense, generalizes the Fourier transform method. Their efforts have provided a significant new tool for mathematical physicists, with applicability far beyond just the Korteweg-de Vries equation.

EXERCISE 2.34. Construct a three-soliton solution for $P = \{1, 1.5, 2\}$. What happens if you alter the phases from the nominal $Q = \{0, 0, 0\}$?

So, to recount the major ideas of this brief sojourn through what we know about electromagnetism, Maxwell's equations describe the macroscopic behavior of electromagnetic phenomena. They can be written in several different mathematical representations, each of which harbors the possibility of finding solutions. It is possible to write Maxwell's equations in a form that is manifestly invariant to Lorentz transform, meaning that they are consistent with Einstein's special theory of relativity. They have formed the basis of modern technology and are quite useful from an engineering perspective. If we begin to drill down into the microscopic world, though, we are faced with the same problem of granularity that we mentioned in our discussion of sand dunes.

The constitutive relations $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$ are the macroscopic properties that arise from the ensemble average over many individual photons. Solutions to Maxwell's equations are additive: if $\mathbf{E}_1$ and $\mathbf{E}_2$ are solutions, then $\mathbf{E}_1 + \mathbf{E}_2$ is a solution. There are many radio stations broadcasting simultaneously: their signals can be detected and processed independently. As a result, we do not have a classical description of a photon. That is a construct that arises only when we take it in context with charged particles. This is a point to which we shall return subsequently.



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